Possible worlds

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A Possible World

- We can think of a possible world as a consistent collection of propositions. The collection must be consistent since under the ordinary rules of logic an inconsistent collection would entail every proposition.
- The notions of “necessity” and “possibility” can be examined by considering the accessibility relations between one designated world and the other worlds.

Accessibility

- If the accessibility relation is reflexive, then the designated world has access to its own propositions.
- If the accessibility relation is symmetric, then if possible world A has access to possible world B, then B has access to A.
- If the accessibility relation is transitive then if world a has access to world B and world B has access to world C, then world A has access to world C.

Definition

- Allowing for different combinations of these accessibility relations provides a foundation for different modal logics.
- With these ideas in mind, we can say the proposition is possible relative to world `H`, if it is true in some world, `Wn`, that is accessible form `H`.
- Likewise a proposition is necessary if it is true in every world, `W0-n`, that is accessible from `H`.

Not Truth-Functional

- In propositional logic, validity can be defined using truth tables. A valid argument is simply one where every truth table row that makes its premises true also makes its conclusion true. However truth tables cannot be used to provide an account of validity in modal logics because there are no truth tables for expressions such as ‘it is necessary that’, ‘it is obligatory that’, and the like.

Valuation

- In propositional logic, a valuation of the atomic sentences (or row of a truth table) assigns a truth-value (T or F) to each propositional variable p. Then the truth-values of the complex sentences are calculated with truth tables.
- In modal semantics, a set W of possible worlds is introduced. A valuation then gives a truth-value to each propositional variable *for each of the possible worlds* in W. This means that value assigned to p for world w may differ from the value assigned to p for another world w′.

Basic Interpretations

- The truth-value of the atomic sentence p at world w given by the valuation v may be written `v(p, w)`. Given this notation, the truth values (T for true, F for false) of complex sentences of modal logic for a given valuation `v` (and member `w` of the set of worlds `W`) may be defined by the following truth clauses. ('iff' abbreviates 'if and only if'.)

(¬) `v(¬A, w)=T iff v(A, w)=F`.

(→) `v(A→B, w)=T iff  v(A, w)=F or v(B, w)=T`.

(5) v(***N***A, w)=T iff for every world w′ in W, v(A, w′)=T.

Relation to Quantification

- Clauses ($\neg$) and ($\rightarrow$) simply describe the standard truth table behavior for negation and material implication respectively.
- According to (5), ***N***A is true (at a world w) exactly when A is true in *all* possible worlds. Given the definition of ***P***, (namely, ***P***A = $\neg$***N***$\neg$A) the truth condition (5) insures that ***P***A is true just in case A is true in *some* possible world. Since the truth clauses for ***N*** and ***P*** involve the quantifiers 'all' and 'some' (respectively), the parallels in logical behavior between ***N*** and $\forall$x, and between ***P*** and $\exists$x is as expected.

Validity

- Clauses (¬), (→), and (5) allow us to calculate the truth-value of any sentence at any world on a given valuation. An argument is *5-valid for a given set W* (of possible worlds) if and only if every valuation of the atomic sentences that assigns the premises T at a world in W also assigns the conclusion T at the same world. An argument is said to be *5-valid* iff it is valid for every non empty set of W of possible worlds.
- It has been shown that S5 is sound and complete for 5-validity.