Coalitional Game

GT

(N, v)

N - Finite set of players

v - characteristic function: $2^N \to \mathbb{R}$,

$\forall S \subseteq N$, $v(S)$ is a real-valued payoff denoting "worth" of a coalition.

QUESTIONS

1. Which coalitions are worthwhile?
2. How can payoffs be divided fairly?

e.g. 1 Voting: A Parliament with Parties A, B, C, D has 45, 25, 15, 15 members respectively.

51 votes are needed to pass a bill, therefore, coalitions are needed. The bill is worth \$100 M.

Properties: Super-additive ①

$\forall S, T \subset N$, if $S \cap T = \phi$ then

$$v(S \cup T) \geq v(S) + v(T)$$

i.e. The grand coalition has the highest payoff among all coalition structures as in voting example.

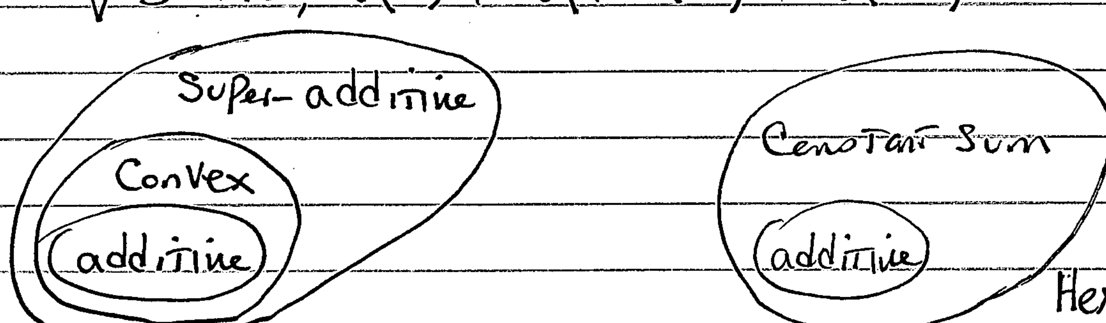
additive (i.e., inessential) ②

$\forall S, T \subset N$, if $S \cap T = \phi$ then

$$v(S \cup T) = v(S) + v(T)$$

Constant Sum ③

$\forall S \subseteq N$, $v(S) + v(N \setminus S) = v(N)$



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A game (N, v) is Convex ④, GT

if $\forall S, T \subset N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$

A game (N, v) is simple ⑤,

if $\forall S \subset N$, $v(S) \in \{0, 1\}$

For a simple game, if $v(S) = 1$, then $\forall T \supset S$, $v(T) = 1$

Proper simple game is a simple game that is constant sum.

e.g. 2

S	$v(S)$
(1)	2
(2)	2
(3)	4
(12)	5
(13)	7
(23)	8
(123)	9

An outcome $\vec{u}$ is feasible if there exists a set of disjoint coalitions whose values are as much as in $\vec{u}$, so we can payoff $\vec{u}$ with v. The disjoint set is a coalition structure. $A = \cup S$

i.e. $\sum_{i \in A} \vec{u}_i = v(A)$

$\vec{u}_1 = \underbrace{\{5, 5, 5\}}_{=15}$ is not feasible since there is no way to divide the agents into subsets they can all get their utility.

$\vec{u}_2 = \underbrace{\{2, 4, 3\}}_{=9}$ is feasible because

$\underbrace{(1)(23)}_{9}$, $\underbrace{(2)(13)}_{9}$, $\underbrace{(123)}_{9}$ can satisfy it.

Problem in $\vec{u}_2$ is that $v((3)) = 4$

That is agent 3 could do better by itself than with a subset. 3 can defect!

there $\vec{u}_2$ is unstable.

The Core, An outcome $\vec{u}$ is stable if no subset can get paid more than they get when they are in $\vec{u}$.

i.e. $\forall S \subset A : \sum_{i \in S} \vec{u}_i \geq v(S)$.

Also, $\vec{u}$ must be feasible.

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e.g. 3

S	N(S)
(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

$\vec{u}_1 = \underbrace{\{2,2,2\}}_{=6}$ is in the core because it is feasible and there is no subset of agents S with a N(S) that is bigger than what they get in $\vec{u}_1$.

$\vec{u}_2 = \underbrace{\{2,2,3\}}_{=7}$ is not in the core because it is not feasible. No coalition structure adds up to 7.

$\vec{u}_3 = \underbrace{\{1,2,2\}}_{=5}$ is not in the core because agents 1 and 2 get 3 from u_3, but they could get 4 since $N((12)) = 4$

Any solution that is in the core cannot be improved. It is Stable!

Here, $\forall\, \vec{u} = \{x, y, z\}$, if $x+y+z \leq 6$ is in the core.

e.g. 4

S	N(S)
(1)	0
(2)	0
(3)	0
(12)	10
(13)	10
(23)	10
(123)	10

e.g. 5

S	N(S)
( )	0
(1)	1
(2)	3
(12)	6

How can we divide 6 in (12)?

$\frac{6}{2} = 3$ is unfair since $N((2)) = 3$.

Why should agent 2 be in (12)?

L. Shapley suggest that each agent should get an amount that corresponds to its marginal contribution to the final value. I.e., the difference between the value before the agent joined the coalition and after he joined. GT

There are $n!$ possibility of joining order, so we can average over orderings.

Let $B(\pi, i)$ be the set of agents in the agent ordering π which appear before agent i. Shapley value for agent i given A agents is given

$$\phi(A, i) = \frac{1}{A!} \sum_{\pi \in \Pi_A} v(B(\pi, i) \cup i) - v(B(\pi, i))$$

$$= \sum_{S \subseteq A} \frac{(|A| - |S|)!\,(|S| - i)!}{|A|!} \left[v(S) - v(S - \{i\})\right]$$

In e.g. 5, π is (12) (21) ← ordering of agents.

$$\phi(\{1,2\}, 1) = \frac{1}{2}\left(v(1) - v() + v(21) - v(2)\right)$$

$$= \frac{1}{2}(1 - 0 + 6 - 3) = 2$$

$$\phi(\{1,2\}, 2) = \frac{1}{2}\left(v(12) - v(1) + v(2) - v()\right)$$

$$= \frac{1}{2}(6 - 1 + 3 - 0) = 4$$

Shapley value always exists & it is unique!

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For e.g. 1,

$$\phi_A = (3)\frac{(4-1)!(2-1)!}{4!}(100-100)$$

$$= 3\frac{2}{24}(100) + (3)\frac{(1)(2)}{24}(100-100)$$

$$= 25 + 25 = \$50 \text{ million}$$

$$\phi_B = \frac{(4-2)!\ (2-1)!}{4!}(100-0) + \frac{(4-3)!\ (3-1)!}{4!}(100-0)$$

$$= \$16.66 \text{ million}$$

Shapley value is computationally expensive. It requires to calculate at least $2^{|A|}$ orderings. This is not realistic.

Shapley does not give coalition structures.

The Core is often empty. We can seek an outcome that minimizes temptation to flee coalitions. That is the ~~nucleas~~ Nucleas.

The excess of coalition S given outcome $\vec{u}$ is given by

$$e(S, \vec{u}) = v(S) - \vec{u}(S),$$

where $\vec{u}(S) = \sum_{i \in S} \vec{u}_i$

i.e., If an outcome $\vec{u}$ is in the Core, all coalitions have an excess less than or equal zero.

Excess-ordered list of coalitions

$$\Theta(\vec{u}) = \langle e(S_1^{\vec{u}}, \vec{u}) \cdots e(S_{2^{|A|}}^{\vec{u}}, \vec{u}) \rangle$$

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lexicograph order outcomes.

e.g. $\{(2,2,2), (2,1,0), (3,2,2), (2,1,1)\} \longrightarrow$

$$\{(3,2,2), (2,2,2), (2,1,1), (2,1,0)\}$$

Nucleaus outcome $\vec{u}$ is one that is not lexicographically bigger than anyone else.

i.e. $\{\vec{u} \mid \theta(\vec{u}) \not> \theta(\vec{v}) \ \forall v \neq \vec{u}$, provided $\vec{u}$ & $\vec{v}$ are both feasible$\}$

Such a $\vec{u}$ is a minimal excess outcome, that is outcomes with the least temptation to flee.

Instead of Shapley value, we could divide payoffs with equal excess.

$$\underbrace{A^{\tau}(i,S)}_{\uparrow} = \max_{T \neq S} \underbrace{E^{\tau}(i,T)}_{\uparrow}$$

i's expected payoff at time τ from not choosing S,
i.e. i's alternate choice.

i's expected payoff for choosing coalition T

$$E^{(\tau+1)}(i,s) = A^{\tau}(i,s) + \frac{v(S) - \sum_{j \in S} A^{\tau}(j,s)}{|s|}$$

$\uparrow$
next iteration of i's expectation

For e.g. 5,

② $E(1,(1)) = 1$ ③ $E^1(1,(12)) = 3$ ④ $A^1(1,(1)) = 3$

① $E^0(2,*) = 0$ $E^1(1,(1)) = 1$ ~~$A^1(1,(1)) = 3$~~

⑤ $A^1(1,(12)) = 1$

⑥ $E^1(2,(2)) = 3$ ⑦ $E^1(2,(12)) = 3$ ⑧ $A^1(1,(1)) = 3$

There are some convergence problems... ⑨ $A^1(1,(12)) = 3$

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Coalition Formation Algm

1. $L_i \leftarrow$ set all coalitions that include i.
2. $S_i^* \leftarrow \arg\max_{S \in L_i} N_i(S)$
3. Broadcast S_i^* and wait for all other broadcasts, put these into S^*_{set}
4. $S_{max} \leftarrow \arg\max_{S \in S^*} N_i(S)$
5. if $i \in S_{max}$
6. then Join S_{max}
7. return
8. for $j \in S_{max}$
9. do Delete all coalitions in L_i that contain j
10. if L_i is not empty
11. then goto 2
12. return

the # of coalition structures = Stirling #.

$$\text{Stirling } \cancel{\#}(A,2) = \frac{1}{2}\sum_{i=1}^{1}\binom{2}{i}(2-i)^A = 2^{A-1}-1.$$